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Impulsively Excited Disturbances in Non-Uniform Boundary Layers

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Abstract

Results will be reviewed for the linearized disturbance impulse response of non-uniform boundary layers. Two distinct forms of boundary layer non-uniformity have been studied. First, we consider the global behaviour of impulsively excited disturbances in temporally steady rotating-disc boundary layers, where there is a spatial inhomogeneity which stems from the radially increasing circumferential velocity. We then consider the oscillatory Stokes layer that is driven by the time-periodic in-plane motion of a bounding flat plate. This provides a second type of boundary layer non-uniformity, which allows us to address the effects of base-flow unsteadiness upon the global development of disturbances.

Introduction

The hydrodynamic stability of viscous flows over solid surfaces has been the subject of a vast number of theoretical investigations [18], ever since Prandtl first formulated the notion of a boundary layer, more than a hundred years ago. These studies have mainly been conducted in order to gain a better understanding of the various transition processes through which a laminar boundary layer may become turbulent. The prediction and control of the onset location for turbulence is important in a wide variety of technological applications. For example, in flows over aircraft wings, there is typically a very large increase in the skin friction, once the boundary layer becomes turbulent. Attempts to find some means of mitigating the detrimental effect upon the overall drag, either through delaying or completely avoiding transition, have recently gained an added urgency. This is because of the need to reduce the fuel burnt by aircraft, in order to diminish the associated emissions of carbon dioxide.

Much attention has been focused on determining the hydrodynamic stability behaviour for high Reynolds number boundary layer flow configurations, where the underlying base flow is at most only slowly varying along directions that are aligned parallel to the solid surface which bounds the fluid. This allows a so-called parallel flow approximation to be applied, which amounts to treating the base flow as if it were homogeneous with respect to two of the spatial co-ordinates. Any spatial development that is exhibited by the genuine base flow is then only accounted for via an artificially introduced form of parametrization, typically through the introduction of a locally defined Reynolds number. The advantage to be gained from treating the flow in such a manner is that the governing equations for the development of disturbances can be subjected to a Fourier transformation in all but one of three spatial directions. This leads to a considerable simplification of the linear stability problem. It facilitates a categorization of the stability behaviour using eigenvalues that are determined from the solution of ordinary differential equations, avoiding the need to directly tackle the system of partial differential equations that would otherwise arise.

Having treated the spatial variation of the base flow in such an artificial manner, in effect by replacing the genuine stability problem with a parameterized set of much simpler problems, there is the question of how to reassemble the solutions that are thus obtained. This is necessary for the determination of the global stability behaviour of disturbances to the flow. In many cases, the results obtained using the flow homogenisation approximation can be pasted together in a relatively straightforward manner, in order to gain a good indication of the possibilities for the disturbance behaviour. For example, it is well-known that this can readily be done to achieve an acceptably accurate description of the evolution of disturbances in Blasius flow over a flat plate [7], where the genuine flow has a spatial inhomogeneity associated with the growing boundary layer thickness. By contrast, we shall illustrate cases of disturbance development where the connection between the behaviour predicted using a flow homogenisation and that which is found in the actual inhomogeneous flow is much less immediate and obvious. In the absence of any further investigation, involving a more careful analysis, some of the results that we will discuss may seem to display a somewhat counter-intuitive character.

After we have shown that the spatial inhomogeneity of a boundary layer can sometimes lead to forms of disturbance behaviour that are rather different from what might have been naively anticipated, we will then move on to briefly illustrate how some equally unexpected effects can be found to arise when non-uniformity in space is replaced by a non-uniformity in time. In both cases, we will focus on the linear impulse response for disturbances, and present the results of numerical simulations [3, 2] that were conducted without resort to any form of artificial simplification of the base flow.

Disturbance Evolution in Rotating-Disc Boundary Layers

The three-dimensional von Kármán rotating-disc boundary layer is formed when a flat solid surface rotates beneath an otherwise stationary incompressible viscous fluid [11]. For more than half a century [8], it has been the subject of stability investigations, which have become increasingly refined [16]. This prolonged attention from the fluid dynamics research community can partly be attributed to the relative simplicity of the boundary layer, which may be described using a similarity solution that is readily computed. But interest has also been maintained because the flow is susceptible to the same kind of crossflow instability that is found in swept wing boundary layer configurations, which have a more immediate practical relevance.

The exact von Kármán solution has radial and azimuthal velocity components that grow linearly with the radial distance from the axis rotation, thus reflecting the increasing circumferential speed of the disc surface. However, most of the hydrodynamic stability studies that were previously conducted made use of a flow approximation in which the flow was taken to be homogeneous along the radial addition. As was outlined in the introduction, this affords a simplification of the linear stability problem, but at the cost of demoting the radial dependence of the genuine flow to the role of a parameter. In this particular case, the parameter can be taken to be the non-dimensional radius at which the flow homogenization approximation is applied, which may also be construed as a locally defined Reynolds number.

Lingwood [13] showed that, according to a stability analysis conducted using the flow homogenization, an absolute instability sets in at a particular value of the non-dimensional radius. Moreover, once this value is exceeded, the temporal growth rate associated with the instability gets larger and larger with the radius. Though the radius is treated merely as a parameter in the locally simplified form of the stability analysis, it was presumed that the increasing strength that was indicated for the absolute instability ought to be sufficient to ensure that a global form of unstable behaviour would arise in the genuine flow. This expectation was confounded by the numerical simulation results reported by Davies and Carpenter [4]. These showed that no modes of instability that exhibited a global temporal growth could be detected in the genuine flow, at least not for any forms of disturbance that remained small enough to be accounted for using a linearisation. It was thus unexpectedly discovered that the flow remains globally stable, when the radial inhomogeneity of the base flow is properly incorporated, dropping the approximation that imposed a radial homogenization. Initially this finding was taken to be somewhat counter-intuitive. It was unclear how a flow that could be classified as becoming increasingly unstable, when viewed using the parametric characterization of the radial position that is deployed within the homogenized approximation, would as yet remain globally stable.

An explanation for such unexpected behaviour was later afforded by considering the radial variation of the temporal frequency which is associated with the absolute instability [6]. This was found to change sufficiently rapidly to ensure that no globally dominant frequency could be selected for the evolution of the disturbances. The global stability was thus attributed to the effect of what was labelled as the *detuning* of the absolute instability frequency. Subsequent studies, conducted for modifications of the von Kármán boundary layer that incorporated mass suction and injection at the disc surface, as well as an axial magnetic field, demonstrated that the occurrence of significant global stability effects arising from temporal frequency detuning were a generic feature for a wide range of three-dimensional rotating flows. However, for cases where the boundary layer was subjected to a sufficiently high level of surface suction, or a strong enough axial magnetic field, it was discovered that the detuning could be construed as being merely an ingredient in the promotion of global instability, rather than having a stabilizing effect [20, 21]. In fact, a novel form of faster than exponential disturbance growth could be triggered. Recent physical experiments for a rotating disc with surface suction, which were carefully conducted by Ho, Corke and Matlis [10], appear to confirm the existence of such behaviour. Thus, although suction at the disk surface is well-known to be locally stabilizing [9], with a relatively modest level of suction doubling the critical Reynolds number for the onset of absolute instability [15], it turns out that it can also give rise to a very markedly detrimental effect upon the global stability.

The numerical simulation studies of the disturbance development in rotating-disc boundary layers have recently been extended [5] to include other members of the so-called Bödewadt-Ekman-Kármán (BEK) family of flows. This brings into consideration a broader class of configurations, where the fluid that lies above the surface of the disc is also allowed to rotate at a non-zero constant rate beyond the boundary layer.

Figure 1 shows the spatial-temporal development of an impulsively excited disturbance for a particular example of a BEK rotating flow, with the Rossby number chosen as $Ro = 0.4$. This represents a case where the rotation of the fluid at large distances away from the disc is more rapid than that of the disc itself, which tends to promote the radially inward convection of disturbances. The contours displayed in the uppermost of the two plots were obtained from a simulation that was conducted using an artificially homogenised base flow. Behaviour that is consistent with the onset of absolute instability [14] can be discerned, by noting that for sufficiently late times, there is a near vertical orientation of the trailing edge of the disturbance, above the point where the impulsive excitation was applied. The response that is found for the genuine radially inhomogeneous flow is shown in the lower plot. This can be seen to be very markedly different. For the earlier times following after the impulse, both edges of the disturbance wavepacket propagate radially inwards. However, for a radius and a time that can be identified towards the top left hand corner of the spatial-temporal region that is depicted, the leading edge reverses direction and

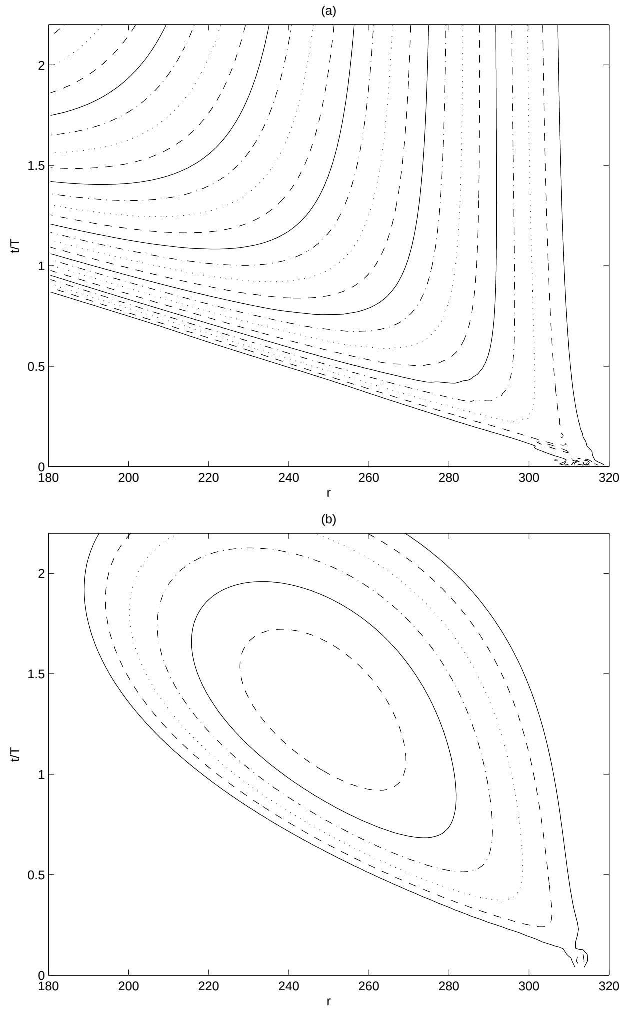

Figure 1: Spatial-temporal contours of the azimuthal component of the perturbation vorticity at the disc surface for an impulsively excited disturbance in a BEK rotating boundary layer. The Rossby number is $Ro = 0.4$ and the azimuthal mode number $n = 50$. (a) Wavepacket development in the artificially homogenized version of the flow; (b) Corresponding evolution in the genuine, radially inhomogeneous flow. The impulse was centred at equivalent radial positions in each case. The contours are plotted using a logarithmic scaling of the amplitudes.

begins to move radially outwards. Observation of the inner contours, for the larger disturbance amplitudes that are contained within the wavepacket, strongly suggests that the two edges will eventually re-connect. Thus, the region where the disturbance is displaying unstable behaviour would appear to be bounded in both space and time. For late enough times, the disturbance will have decayed at every radial location and the flow returned to a quiescent state.

The occurrence of this kind of behaviour cannot be anticipated by naively putting together the results that are obtained using a linear stability analysis based upon an homogeneous flow approximation. As was the case for the von Kármán boundary layer, such an analysis indicates that the temporal growth rate associated with the absolute instability increases with the radius. Despite what this increase in the locally determined measure of the instability might be thought to suggest, the qualitative form of the globally stable behaviour that is depicted in figure 1b was found to persist even when the disturbance was impulsively excited at a larger radius, chosen to lie well within the region where absolute instability was expected. However, the seemingly counter-intuitive nature of the discovered disturbance development may still be explained in a satisfactory manner, provided that the effects of the temporal frequency detuning are appropriately taken into consideration.

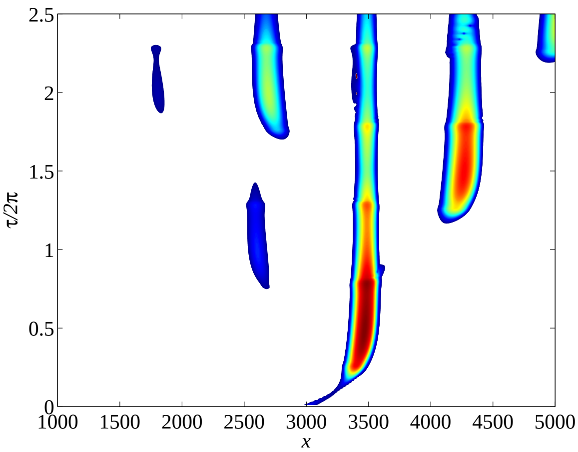

Figure 2: Spatial-temporal contours of the perturbation vorticity at the wall surface for an impulsively disturbance in the oscillatory Stokes boundary layer. The Reynolds number is $Re = 650$. The contours are plotted using a logarithmic scaling of the amplitudes. A lower cut-off level has been applied in order to more easily identify the regions where the flow has been disturbed.

Family-Tree Structure in the Oscillatory Stokes Layer

The second type of boundary layer that we will consider allows us to examine some intriguing effects of base-flow unsteadiness on the global development of disturbances. We will confine our attention to flows with a periodic form for the temporal dependency, which is arguably the simplest case that can be physically realised. This offers a contrast to the type of the spatial dependence that was studied for the rotating-disc boundary layers, where the basic state exhibits a linear variation with the radius.

Numerical simulations [22] were conducted for the Stokes layer that is created by the time-periodic in-plane motion of a flat plate, which bounds what would otherwise be a stationary body of incompressible fluid. This configuration may be construed as providing the archetypal case of an oscillatory boundary layer.

The unsteadiness of the basic state was unexpectedly found to give rise to multiple wavepackets for the impulse response, which displayed an intricate family-tree-like spatial-temporal structure. Such behaviour is illustrated in figure 2. The disturbance was excited by imposing a forcing at the wall, over a small range of spatial positions located near to the centre of the depicted computational domain, during a very short time interval. For early enough times, following on from the application of the impulse, there is just a single wavepacket, which propagates towards the right, matching the direction that was chosen for the initial phase of the oscillatory base-flow motion near to the wall. This is exactly as would be the case for a disturbance developing in a steady boundary layer. However, at the later times the situation becomes more complex. It may be seen that the original wavepacket has somehow triggered the formation of a number of other distinct wavepackets.

A careful analysis of the simulation data revealed that these wavepackets may be classified as *daughter* and *granddaughter* wavepackets that, at particular phases of the oscillation cycle, are born in successive generations from the initial *mother* wavepacket. They eventually become separated from each other by distances that are very much larger than the thickness of the boundary layer. The separations are also large in comparison with the wavelengths which characterise the spatially oscillatory behaviour that is found within each of the individual wavepackets. It may thus be noted that, in order to avoid the introduction of any artificial disturbance reflection effects, it proved necessary to make use of a computational domain with a far greater spatial extent than might otherwise have been thought to be appropriate.

A linear stability analysis [1], conducted for disturbances with a Floquet mode form [12], can be used to predict some of the important features of the simulated spatial-temporal evolution. In particular, it is possible to anticipate the asymptotic temporal growth rate of the maximum of the disturbance amplitude, as well as the most dominant wavenumber exhibited in the spatial spectra. However, the family-tree-like structure that was revealed in the numerical simulations was completely unexpected. Methods for predicting and understanding its overall character are still a matter of active investigation. It would seem that the modelling that was previously used to account for the effects of spatial inhomogeneity, for the case of rotating disk boundary layers, cannot be readily adapted to explain the more complicated disturbance development that arises when a base-flow unsteadiness is introduced.

It has recently been discovered [17] that absolute instability can be promoted by adding a low-amplitude background of noise, in the form of high frequency harmonics to the oscillation of the bounding wall. The addition of such noise had already been shown to promote instability for spatially monochromatic disturbances, taking the form of Floquet modes [19]. Nevertheless, it is not yet clear why it can also trigger the global temporal growth of disturbances that develop from a spatially localised impulse. No such behaviour had previously been identified for the family-tree-like disturbance structures that were found when no higher harmonics were present in the oscillatory wall motion.

Concluding Remarks

For our two chosen non-uniform boundary layer configurations, we have provided a brief discussion of some interesting patterns of behaviour that have been discovered only relatively recently. The aim has been to provide an overview and to highlight some of the novel features. These had not been anticipated from the results of previous studies for simpler configurations, in which a steady in time boundary layer was commonly treated, while being subjected to an approximation of spatial homogeneity along the dominant direction for the disturbance propagation.

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References

[1] Blennerhassett, P. J. and Bassom, A. P., The linear stability of flat Stokes layers, *J. Fluid Mech.*, **464**, 2002, 393–410.

[2] Davies, C., Numerical simulation of boundary-layer disturbance evolution, *Phil. Trans. R. Soc. A* **363**, 2005, 1109–1118.

[3] Davies, C. and Carpenter, P. W., A novel velocity-vorticity formulation of the Navier-Stokes equations with applications to boundary layer disturbance evolution, *J. Comput. Phys.*, **172**, 2001, 119–165.

[4] Davies, C. and Carpenter, P. W., Global behaviour corresponding to the absolute instability of the rotating-disc boundary layer, *J. Fluid Mech.*, **486**, 2003, 287–329.

[5] Davies, C. and Thomas, C., Global stability behaviour for the BEK family of rotating boundary layers, *Theoretical and Computational Fluid Dynamics*, 2016, (in press).

[6] Davies, C., Thomas, C. and Carpenter, P. W., Global stability of the rotating-disc boundary layer, *J. Eng. Math.*, **57**, 2007, 219–236.

[7] Gaster, M., A Theoretical Model of a Wave Packet in the Boundary Layer on a Flat Plate, *Proc. R. Soc. A*, **347**, 1975, 271–289.

[8] Gregory, N., Stuart, J. T. and Walker, W. S., On the Stability of Three-Dimensional Boundary Layers with Application to the Flow due to a Rotating Disk, *Phil. Trans. R. Soc. A*, **248**, 1955, 155–199.

[9] Gregory, N. and Walker, W. S., Experiments on the effect of suction on the flow due to a rotating disk., *J. Fluid Mech.*, **9**, 1960, 225–234.

[10] Ho, J., Corke, T.C. and Matlis, E., Effect of wall suction on rotating disk absolute instability, *J. Fluid Mech.*, **791** , 2016, 704–737.

[11] von Kármán, Th., Über laminare und turbulente Reibung, *Z. Angew. Math. Mech.*, **1**, 1921, 233–252.

[12] von Kerczek, C. and Davis, S. H., Linear stability theory of oscillatory Stokes layers, *J. Fluid Mech*, **62**, 1974, 753–773.

[13] Lingwood, R. J., Absolute instability of the boundary layer on a rotating disk, *J. Fluid Mech.*, **299**, 1995, 17–33.

[14] Lingwood, R. J., Absolute instability of the Ekman layer and related rotating flows, *J. Fluid Mech.*, **331**, 1997, 405-428.

[15] Lingwood, R. J., On the effects of suction and injection on the absolute instability of the rotating-disk boundary layers, *Phys. Fluids*, **9**, 1997, 1317–1328.

[16] Lingwood, R.J. and Alfredsson, P.H., Instability of the von Kármán boundary layer, *Appl. Mech. Rev.*, **67**, 2015, 030803.

[17] Ramage, A., Davies, C. and Thomas, C., Onset of absolute instability in oscillatory Stokes layers, *11th European Fluid Mechanics Conference*, 2016.

[18] Schmid, P.J. and Henningson, D.S., Stability and Transition in Shear Flows, Springer, 2001.

[19] Thomas, C., Blennerhassett, P.J., Bassom, A.P. and Davies, C. The linear stability of a Stokes layer subjected to high-frequency perturbations. *J. Fluid Mech.*, **764**, 2015, 193–218.

[20] Thomas, C. and Davies, C., The effects of mass transfer on the global stability of the rotating-disk boundary layer, *J. Fluid Mech.*, **663**, 2010, 401–433.

[21] Thomas, C. and Davies, C., Global stability of the rotating-disc boundary layer with an axial magnetic field, *J. Fluid Mech.*, **724**, 2013, 510–526.

[22] Thomas, C., Davies, C., Bassom, A.P. and Blennerhassett, P.J. Evolution of disturbance wavepackets in the oscillatory Stokes layer, *J. Fluid Mech.*, **752**, 2014, 543–571.